ON CUSPIDAL WEYL GROUPS AND CUSPIDAL ARTIN GROUPS (Aspects of Mirror Symmetry)

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1. Introduction

We associate a generalized root system in the sense of Kyoji Saito to an orbifold projective line via the derived category of finite dimensional representations of a certain bound quiver algebra. We generalize results by Saito–Takebayshi [28] and Yamada [33] for elliptic Weyl groups and elliptic Artin groups to the Weyl groups and the fundamental groups of the regular orbit spaces associated to the generalized root systems. Moreover we study the relation between this fundamental group and a certain subgroup of the autoequivalence group of a triangulated subcategory of the derived category of 2-Calabi–Yau completion of the bound quiver algebra.

This report is a brief summary of the joint work with Atsushi Takahashi and Kentaro Wada [31]. For precise proofs and the relation of our results to mirror symmetry, see [31] and the report written by Atsushi Takahashi.

2. Notations and terminologies

Throughout this paper, k denotes an algebraically closed field of characteristic zero.

2.1. Generalized root systems.

In this subsection, we recall the definition of the simply laced generalized root system introduced by K. Saito [25, 27].

Definition 2.1. A *simply-laced generalized root system* R consists of

- a free $\mathbb{Z}$-module $K_0(R)$ of finite rank $(=: \mu)$ called the *root lattice*,
- a symmetric bi-linear form $I_R : K_0(R) \times K_0(R) \longrightarrow \mathbb{Z}$,
- a subset $\Delta_{re}(R)$ of $K_0(R)$ called the *set of real roots* such that:
  - (i) $K_0(R) = \mathbb{Z}\Delta_{re}(R)$,
  - (ii) For all $\alpha \in \Delta_{re}(R)$, $I(\alpha, \alpha) = 2$,
  - (iii) For all $\alpha \in \Delta_{re}(R)$, the element r_α of $\mathrm{Aut}(K_0(R), I_R)$, the group of automorphisms of $K_0(R)$ respecting I_R, defined by

$$r_\alpha(\lambda) := \lambda - I_R(\lambda, \alpha)\alpha, \quad \lambda \in K_0(R), \tag{2.1}$$

    makes $\Delta_{re}(R)$ invariant, namely, $r_\alpha(\Delta_{re}(R)) = \Delta_{re}(R)$,

(iv) Let $W(R)$ be the *Weyl group* of R defined by

$$W(R) := \langle r_\alpha \mid \alpha \in \Delta_{re}(R) \rangle \subset \mathrm{Aut}(K_0(R), I_R). \tag{2.2}$$

Then there exists a subset $B = \{\alpha_1, \ldots, \alpha_\mu\}$ of $\Delta_{re}(R)$ called a *root basis* of R which satisfies $K_0(R) = \bigoplus_{i=1}^{\mu} \mathbb{Z}\alpha_i$, $W(R) = \langle r_{\alpha_1}, \ldots, r_{\alpha_\mu} \rangle$ and $\Delta_{re}(R) = W(R)B$.

- an element c_R of $W(R)$ called the *Coxeter transformation*, which has the product presentation $c_R = r_{\alpha_1} \cdots r_{\alpha_\mu}$ with respect to some root basis $B = \{\alpha_1, \ldots, \alpha_\mu\}$.

An element of $\Delta_{re}(R)$ is called a *real root* and an element of B is called a *real simple root*. For a real simple root $\alpha \in B$, the reflection r_α is called a *simple reflection*.

Definition 2.2. Let $R = (K_0(R), I_R, \Delta_{re}(R), c_R)$ be a simply-laced generalized root system with a root basis $B = \{\alpha_1, \ldots, \alpha_\mu\}$ of R. The *Coxeter–Dynkin diagram* Γ_B is a finite graph defined as follows:

- the set of vertices is $B = \{\alpha_1, \ldots, \alpha_\mu\}$,
- the edge between vertices α_i and α_j of Γ_B is given by the following rule:

$$\circ_{\alpha_i} \qquad \circ_{\alpha_j} \quad \text{if } \ I_R(\alpha_i, \alpha_j) = 0, \tag{2.3a}$$

$$\circ_{\alpha_i} \text{———} \circ_{\alpha_j} \quad \text{if } \ I_R(\alpha_i, \alpha_j) = -1, \tag{2.3b}$$

$$\circ_{\alpha_i} \underset{t}{\text{———}} \circ_{\alpha_j} \quad \text{if } \ I_R(\alpha_i, \alpha_j) = -t, \ (t \geq 2), \tag{2.3c}$$

$$\circ_{\alpha_i} \cdots\cdots \circ_{\alpha_j} \quad \text{if } \ I_R(\alpha_i, \alpha_j) = +1, \tag{2.3d}$$

$$\circ_{\alpha_i} \ \vdots\vdots\vdots\vdots\vdots\vdots \ \circ_{\alpha_j} \quad \text{if } \ I_R(\alpha_i, \alpha_j) = +2, \tag{2.3e}$$

$$\circ_{\alpha_i} \underset{t}{\cdots\cdots} \circ_{\alpha_j} \quad \text{if } \ I_R(\alpha_i, \alpha_j) = +t, \ (t \geq 3). \tag{2.3f}$$

2.2. **Generalized root systems from triangulated categories.** In this subsection, we deduce a simply laced generalized root system from a certain algebraic triangulated category which satisfies plausible conditions.

Definition 2.3. Let $\mathcal{D}$ be a k-linear triangulated category with the translation functor $[1]$. Consider a free abelian group F with generators $\{[X] \mid X \in \mathcal{D}\}$ and a subgroup F_0 of F generated by $[X] - [Y] + [Z]$ for all exact triangles $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ in $\mathcal{D}$. The *Grothendieck group* $K_0(\mathcal{D})$ of $\mathcal{D}$ is a quotient group F/F_0.

Any triangulated category of our interest in this paper is equipped with an enhancement. We briefly recall some terminologies.

Definition 2.4 ([15]). Let $\mathcal{D}$ be a k-linear triangulated category. We say that $\mathcal{D}$ is *algebraic* if it is equivalent as a triangulated category to the stable category of some k-linear Frobenius category.

It is important to note that for an algebraic k-linear triangulated category $\mathcal{D}$, we have functorial cones and $\mathbb{R}$Hom-complexes once we fix an enhancement, a differential graded category which yields $\mathcal{D}$ (see Theorem 3.8 in [16] for precise statements).

Definition 2.5. Let $\mathcal{D}$ be an algebraic k-linear triangulated category with the translation functor [1] with a fixed enhancement.

(i) For $X, Y \in \mathcal{D}$, denote by $\mathbb{R}\mathrm{Hom}^\bullet_{\mathcal{D}}(X,Y) \in \mathcal{D}(k)$ the $\mathbb{R}$Hom-complex such that $\mathrm{Hom}_{\mathcal{D}}(X, Y[p]) = H^p(\mathbb{R}\mathrm{Hom}^\bullet_{\mathcal{D}}(X,Y))$ for all $p \in \mathbb{Z}$, where $\mathcal{D}(k)$ is the derived category of complexes of k-modules.

(ii) A k-linear triangulated category $\mathcal{D}$ is said to be *of finite type* if the total dimension of the graded k-module $\mathrm{Hom}^\bullet_{\mathcal{D}}(X,Y) := \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(X, Y[p])[-p]$ is finite for all $X, Y \in \mathcal{D}$.

Definition 2.6. Let $\mathcal{D}$ be an algebraic k-linear triangulated category of finite type with a fixed enhancement.

(i) An object E in $\mathcal{D}$ is called an *exceptional object* (or is called *exceptional*) if $\mathbb{R}\mathrm{Hom}^\bullet_{\mathcal{D}}(E,E) \cong k \cdot \mathrm{id}_E$ in $\mathcal{D}(k)$.

(ii) An *exceptional collection* $\mathcal{E} = (E_1, \ldots, E_n)$ in $\mathcal{D}$ is a finite ordered set of exceptional objects satisfying the condition that $\mathbb{R}\mathrm{Hom}^\bullet_{\mathcal{D}}(E_i,E_j) \cong 0$ in $\mathcal{D}(k)$ for all $i > j$. An exceptional collection consisting of two objects is an *exceptional pair.*

(iii) An exceptional collection $\mathcal{E} = (E_1, \ldots, E_n)$ in $\mathcal{D}$ is said to be *isomorphic* to another exceptional collection $\mathcal{E}' = (E'_1, \ldots, E'_n)$ in $\mathcal{D}$ if $E_i \cong E'_i$ in $\mathcal{D}$ for all $i = 1, \ldots, \mu$.

(iv) An exceptional collection $\mathcal{E} = (E_1, \ldots, E_n)$ in $\mathcal{D}$ is called a *strongly exceptional collection* if, for all $i, j = 1, \ldots, n$, the complex $\mathbb{R}\mathrm{Hom}^\bullet_{\mathcal{D}}(E_i,E_j)$ is isomorphic in $\mathcal{D}(k)$ to a complex concentrated in degree zero, equivalently, we have $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[p]) = 0$ for $p \neq 0$.

(v) An exceptional collection $\mathcal{E}$ in $\mathcal{D}$ is called *full* if the smallest full triangulated subcategory of $\mathcal{D}$ containing all elements in $\mathcal{E}$ is equivalent to $\mathcal{D}$.

(vi) For an exceptional pair (X, Y), one has new exceptional pairs $(\mathcal{L}_X Y, X)$ called the *left mutation* of (X, Y) and $(Y, \mathcal{R}_Y X)$ called the *right mutation* of (X, Y). Here the object $\mathcal{L}_X Y[1]$ is defined as the cone of the evaluation morphism ev

$$\mathbb{R}\mathrm{Hom}^\bullet_{\mathcal{D}}(X,Y) \otimes^{\mathbb{L}} X \xrightarrow{ev} Y, \tag{2.4a}$$

where $(-) \otimes^{\mathbb{L}} X$ is the left adjoint of the functor $\mathbb{R}\mathrm{Hom}_{\mathcal{D}}(X, -) : \mathcal{D} \longrightarrow \mathcal{D}(k)$. Similarly, the object $\mathcal{R}_Y X$ is defined as the cone of the evaluation morphism ev^*

$$X \xrightarrow{ev^*} \mathbb{R}\mathrm{Hom}^{\bullet}_{\mathcal{D}}(X, Y)^* \otimes^{\mathbb{L}} Y. \tag{2.4b}$$

where $(-)^*$ denotes the duality $\mathrm{Hom}_k(-, k)$.

Here we recall the braid group action on the set of isomorphism classes of full exceptional collections.

Definition 2.7. The Artin's *braid group* B_μ on μ-strands is a group presented by the following generators and relations:

Generators: $\{b_i \mid i = 1, \dots, \mu - 1\}$

Relations:

$$b_i b_j = b_j b_i \quad \text{for} \quad |i - j| \geq 2, \tag{2.5a}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \quad \text{for} \quad i = 1, \dots, \mu - 2. \tag{2.5b}$$

Consider the group $G_\mu := B_\mu \ltimes \mathbb{Z}^\mu$, the semi-direct product of the braid group B_μ and the free abelian group of rank μ, defined by the group homomorphism $B_\mu \to \mathfrak{S}_\mu \to \mathrm{Aut}_{\mathbb{Z}}\mathbb{Z}^\mu$, where the first homomorphism is $b_i \mapsto (i, i+1)$ and the second one is induced by the natural actions of the symmetric group $\mathfrak{S}_\mu$ on $\mathbb{Z}^\mu$.

Proposition 2.8 (cf. Proposition 2.1 in [3]). *Let $\mathcal{D}$ be an algebraic k-linear triangulated category of finite type with a fixed enhancement.. The group G_μ acts on the set of isomorphism classes of full exceptional collections in $\mathcal{D}$ by mutations and translations:*

$$b_i(E_1, \dots, E_\mu) := (E_1, \dots, E_{i-1}, E_{i+1}, \mathcal{R}_{E_{i+1}} E_i, E_{i+2}, \dots, E_\mu), \tag{2.6a}$$

$$b_i^{-1}(E_1, \dots, E_\mu) := (E_1, \dots, E_{i-1}, \mathcal{L}_{E_i} E_{i+1}, E_i, E_{i+2}, \dots, E_\mu), \tag{2.6b}$$

$$e_i(E_1, \dots, E_\mu) := (E_1, \dots, E_{i-1}, E_i[1], E_{i+1}, \dots, E_\mu), \tag{2.6c}$$

where we denote by e_i the i-th standard basis of $\mathbb{Z}^\mu$. □

Proposition 2.9. *Let $\mathcal{D}$ be an algebraic k-linear triangulated category of finite type with the translation functor $[1]$ and a fixed enhancement. Assume that $\mathcal{D}$ satisfies the following conditions:*

(i) *There exists a full strongly exceptional collection $\mathcal{E} = (E_1, \dots, E_\mu)$ in $\mathcal{D}$.*

(ii) *The action of the group G_μ on the set of isomorphism classes of full exceptional collections in $\mathcal{D}$ is transitive.*

(iii) *For any exceptional object $E' \in \mathcal{D}$, there exists a full exceptional collection $\mathcal{E}'$ in $\mathcal{D}$ such that $E' \in \mathcal{E}'$.*

Then the following quadruple

- *the Grothendieck group* $K_0(\mathcal{D})$ *of* $\mathcal{D}$,
- *the Cartan form* $I_{\mathcal{D}} : K_0(\mathcal{D}) \times K_0(\mathcal{D}) \longrightarrow \mathbb{Z}$;

$$I_{\mathcal{D}}([X],[Y]) := \chi_{\mathcal{D}}([X],[Y]) + \chi_{\mathcal{D}}([Y],[X]), \quad X, Y \in \mathcal{D}, \tag{2.7}$$

where $\chi_{\mathcal{D}} : K_0(\mathcal{D}) \times K_0(\mathcal{D}) \longrightarrow \mathbb{Z}$ *is the Euler form defined by*

$$\chi_{\mathcal{D}}([X],[Y]) := \sum_{p \in \mathbb{Z}} (-1)^p \dim_k \mathrm{Hom}_{\mathcal{D}}(X, Y[p]), \tag{2.8}$$

- *the subset* $\Delta_{re}(\mathcal{D})$ *of* $K_0(\mathcal{D})$ *defines by*

$$\Delta_{re}(\mathcal{D}) := W(B)B, \quad B := \{[E_1], \dots, [E_\mu]\}, \tag{2.9}$$

where $W(B)$ *is a subgroup of* $\mathrm{Aut}(K_0(\mathcal{D}), I_{\mathcal{D}})$ *generated by reflections*

$$r_{[E_i]}(\lambda) := \lambda - I_{\mathcal{D}}(\lambda, [E_i])[E_i], \quad \lambda \in K_0(\mathcal{D}), \quad i = 1, \dots, \mu, \tag{2.10}$$

- *the automorphism* $c_{\mathcal{D}}$ *on* $K_0(\mathcal{D})$ *induced by the Coxeter functor* $\mathcal{C}_{\mathcal{D}} := \mathcal{S}_{\mathcal{D}}[-1]$ *on* $\mathcal{D}$ *where* $\mathcal{S}_{\mathcal{D}}$ *is the Serre functor on* $\mathcal{D}$,

forms a simply-laced generalized root system $R_{\mathcal{D}}$, *which does not depend on the choice of the full exceptional collection* $\mathcal{E}$.

Sketch of Proof. The lattice and the Cartan form are derived invariants. Thus we only have to check the assertion about the set of the real root and the Coxeter element. The following lemma holds from a relation between the Serre functor $\mathcal{S}_{\mathcal{D}}$ on $\mathcal{D}$ and the helix generated by the full exceptional collection $\mathcal{E}$. See p. 223 in [3].

Lemma 2.10. *We have*

$$c_{\mathcal{D}} = r_{[E_1]} \cdots r_{[E_\mu]}. \tag{2.11}$$

By direct calculation, we have the following lemma:

Lemma 2.11. *For any* $\alpha \in \Delta_{re}(\mathcal{D})$, *we have*

$$r_{[E_i]} r_\alpha = r_{r_{[E_i]}(\alpha)} r_{[E_i]}. \tag{2.12}$$

Note that Lemma 2.11 implies that $W(\mathcal{D}) = W(B)$. By Lemma 2.11 and the assumption (ii) and (iii), we have the following lemma:

Lemma 2.12. *For an exceptional object* $E' \in \mathcal{D}$, *the class* $[E'] \in K_0(\mathcal{D})$ *belongs to* $\Delta_{re}(\mathcal{D})$.

Set $B' := \{[E'_1], \dots [E'_\mu]\}$ for any full exceptional collection $\mathcal{E}' = (E'_1, \dots, E'_\mu)$ in $\mathcal{D}$. Lemma 2.12 implies that $W(\mathcal{D})B' \subset W(\mathcal{D})W(\mathcal{D})B \subset W(\mathcal{D})B$ and hence $W(\mathcal{D})B' = W(\mathcal{D})B$. Therefore the set $\Delta_{re}(\mathcal{D})$ does not depend on the particular choice of the full exceptional collection $\mathcal{E}$. □

Remark 2.13. We assumed in Proposition 2.9 the existence of a full *strongly* exceptional collection $\mathcal{E}$ in $\mathcal{D}$ in order to ensure that $\mathcal{D}$ has a unique enhancement in a suitable sense. We refer [14] and [19] for some results on the uniqueness of enhancements for triangulated categories and do not discuss this matter more in detail.

Definition 2.14. The generalized root system $R_{\mathcal{D}}$ in Proposition 2.9 is called the *simply-laced generalized root system associated to $\mathcal{D}$.*

It is natural to expect the assumptions of Proposition 2.9. Indeed, they are proven for derived categories of hereditary Artin algebras by Crawley-Boevey [7] and Ringel [22] and for derived categories of coherent sheaves on an orbifold projective line $\mathbb{P}^1_{A,\Lambda}$ (we shall recall the definition later) by Meltzer [21]. The transitivity of the action of G_μ is conjectured by Bondal–Polishchuk (Conjecture 2.2 in [3]), and is known for the derived categories of coherent sheaves on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ by Rudakov [24], by arbitrary del Pezzo surfaces by Kuleshov and Orlov [18], for example.

Remark 2.15. One can also consider the subset $\Delta^s_{re}(\mathcal{D})$ of $K_0(\mathcal{D})$ defined by

$$\Delta^s_{re}(\mathcal{D}) := \{[E] \in K_0(\mathcal{D}) \mid E \text{ is an exceptional object in } \mathcal{D}\}, \tag{2.13}$$

which is known as the set of *Schur roots*. Under the assumptions of Proposition 2.9, we always have $\Delta^s_{re}(\mathcal{D}) \subset \Delta_{re}(\mathcal{D})$, however, $\Delta^s_{re}(\mathcal{D}) \neq \Delta_{re}(\mathcal{D})$ in general. Criteria to have $\Delta^s_{re}(\mathcal{D})$ in terms of the Weyl group $W(\mathcal{D})$ is recently given by Hubery–Krause [10] for derived categories of hereditary Artin algebras.

2.3. **Generalized root systems associated to star quivers.** We recall the definition of quivers and their path algebras.

Definition 2.16. A *quiver* Q is a quadruple $(Q_0, Q_1; s, t)$ where Q_0 is a set called the set of *vertices*, Q_1 is a set called the set of *arrows* and s, t are maps from Q_1 to Q_0 which associate the *source* vertex and the *target* vertex for each arrow. An arrow f with the source $s(f)$ and the target $t(f)$ is often written as $s(f) \xrightarrow{f} t(f)$.

Definition 2.17. Let $Q = (Q_0, Q_1; s, t)$ be a quiver.

(i) A *path of length* 0 is a symbol $(v|v)$ defined for each vertex $v \in Q_0$.

(ii) A *path of length* $l \geq 1$ from the vertex v to the vertex v' in a quiver Q is a symbol $(v|f_1 \cdots f_l|v')$ with arrows f_i, $i = 1, \ldots, l$ such that $s(f_1) = v$, $t(f_l) = v'$ and $s(f_{i+1}) = t(f_i)$; $i = 1, \ldots, l-1$.

(iii) For a path $p = (v|f_1 \cdots f_l|v')$, set $s(p) := v$ and $t(p) := v'$.

(iv) An ordered pair of paths (p_1, p_2) is *composable* if $t(p_1) = s(p_2)$.

(v) The *composition* of composable paths $((v_1|f_1 \cdots f_l|v_1'), (v_2|g_1 \cdots g_m|v_2'))$ is a path $(v_1|f_1 \cdots f_l g_1 \cdots g_m|v_2')$.

Definition 2.18. Let Q be a quiver.

(i) The *path algebra* kQ of a quiver Q is defined as the k-module generated by all paths in Q together with the associative product structure defined by the composition of paths, where the product of two non-composable paths is set to be zero.

(ii) A *bound quiver* is a pair $(Q, \mathcal{I})$ where Q is a quiver and $\mathcal{I}$ is an ideal of kQ.

(iii) A *bound quiver algebra* $k(Q, \mathcal{I})$ of a bound quiver $(Q, \mathcal{I})$ is defined as the algebra $kQ/\mathcal{I}$.

We recall a special class of quivers called star quivers, which are of our interest.

Definition 2.19. Let $r \geq 3$ be a positive integer and $A = (a_1, \ldots, a_r)$ a tuple of positive integers greater than one. Define a quiver $\mathbb{T}_A = (\mathbb{T}_{A,0}, \mathbb{T}_{A,1}; s, t)$ as follows:

- The set $\mathbb{T}_{A,0}$ of vertices is

$$\mathbb{T}_{A,0} := \{1\} \coprod \left(\coprod_{i=1}^{r} \coprod_{j=1}^{a_i - 1} \{(i,j)\} \right). \tag{2.14a}$$

- The set $\mathbb{T}_{A,1}$ of arrows is

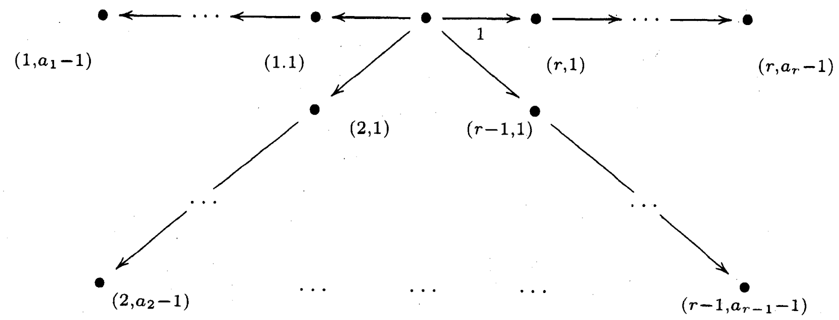
$$\mathbb{T}_{A,1} := \coprod_{i=1}^{r} \coprod_{j=1}^{a_i - 1} \{f_{i,j}\}, \tag{2.14b}$$

whose source $s(f)$ and target $t(f)$ of each arrow f is given as follows;

$$s(f_{i,1}) = 1, \quad t(f_{i,1}) = (i,1), \quad i = 1, \ldots, r, \tag{2.14c}$$

$$s(f_{i,j}) = (i, j-1), \quad t(f_{i,j}) = (i,j), \quad i = 1, \ldots, r, \ j = 1, \ldots, a_i - 1. \tag{2.14d}$$

The quiver $\mathbb{T}_A$ is called the *star quiver of type A*.



Definition 2.20. Let $\mathbb{T}_A$ be a star quiver of type A.

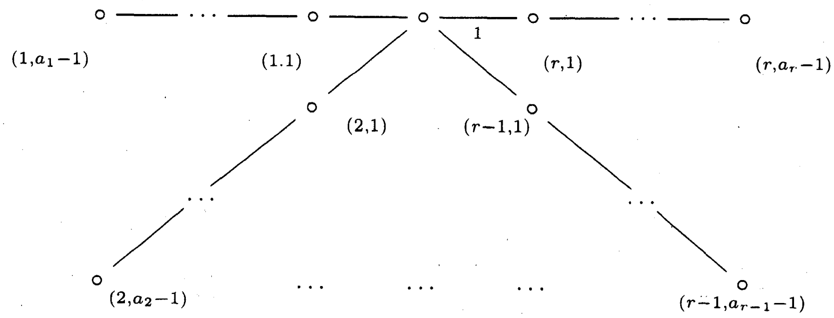
(i) Denote by R_A the generalized root system associated to $\mathcal{D}^b(k\mathbb{T}_A)$.

(ii) Let α_v be the equivalence class in $K_0(R_A) = K_0(\mathcal{D}^b(k\mathbb{T}_A))$ of the simple $k\mathbb{T}_A$-module corresponding to the vertex $v \in \mathbb{T}_{A,0}$. Set

$$B_{\mathbb{T}_A} := \{\alpha_v\}_{v \in \mathbb{T}_{A,0}}, \tag{2.15}$$

which is a root basis of R_A.

(iii) Denote by T_A the Coxeter–Dynkin diagram for $\Gamma_{B_{\mathbb{T}_A}}$, which is given by



We often write $v \in T_A$ instead of $v \in \mathbb{T}_{A,0}$.

(iv) For each $v \in T_A$, define the *simple reflection* r_v on $K_0(R_A)$ by

$$r_v(\lambda) := \lambda - I_{R_A}(\lambda, \alpha_v)\alpha_v, \quad \lambda \in K_0(R_A). \tag{2.16}$$

Since $B_{\mathbb{T}_A}$ is a root basis of R_A, the Weyl group $W(R_A)$ of R_A is generated by simple reflections;

$$W(R_A) = \langle r_v \mid v \in T_A \rangle. \tag{2.17}$$

Note that the Cartan matrix $(I_{R_A}(\alpha_v, \alpha_{v'}))$ is a generalized Cartan matrix in the sense of [12]. Therefore one can naturally associate to R_A a Kac–Moody Lie algebra $\mathfrak{g}(R_A)$.

2.4. **Octopus.** We introduce a bound quiver, a "one point extension" of the star quiver.

Definition 2.21. Let $r \geq 3$ be a positive integer, $A = (a_1, \dots, a_r)$ an r-tuple of positive integers greater than one and $\Lambda = (\lambda_1, \dots, \lambda_r)$ an r-tuple of pairwise distinct elements of $\mathbb{P}^1(k)$ normalized such that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$.

(i) Define a quiver $\widetilde{\mathbb{T}}_A = (\widetilde{\mathbb{T}}_{A,0}, \widetilde{\mathbb{T}}_{A,1}, s, t)$ as follows:

- The set $\widetilde{\mathbb{T}}_{A,0}$ of vertices is given by

$$\widetilde{\mathbb{T}}_{A,0} := \mathbb{T}_{A,0} \coprod \{1^*\} = \{1\} \coprod \left(\coprod_{i=1}^{r} \coprod_{j=1}^{a_i - 1} \{(i,j)\} \right) \coprod \{1^*\}. \tag{2.18a}$$

- The set $\widetilde{\mathbb{T}}_{A,1}$ of arrows is given by

$$\widetilde{\mathbb{T}}_{A,1} := \mathbb{T}_{A,1} \coprod \left(\coprod_{i=1}^{r} \{f_{i,1^*}\} \right) = \left(\coprod_{i=1}^{r} \coprod_{j=1}^{a_i - 1} \{f_{i,j}\} \right) \coprod \left(\coprod_{i=1}^{r} \{f_{i,1^*}\} \right), \tag{2.18b}$$

whose source $s(f)$ and target $t(f)$ of each arrow f is given as follows:

$$s(f_{i,1}) = 1, \quad t(f_{i,1}) = (i,1), \quad i = 1, \dots, r, \tag{2.18c}$$

$$s(f_{i,j}) = (i, j-1), \quad t(f_{i,j}) = (i,j), \quad i = 1, \dots, r, \ j = 2, \dots, a_i - 1, \tag{2.18d}$$

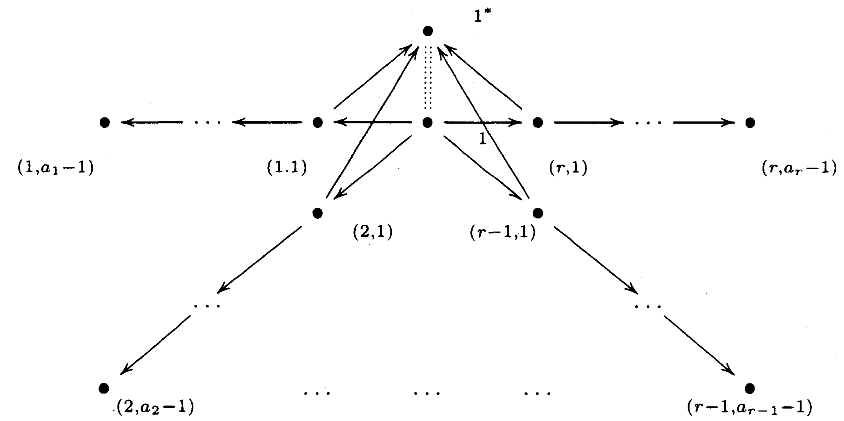
$$s(f_{i,1^*}) = (i,1), \quad t(f_{i,1^*}) = 1^*, \quad i = 1, \dots, r. \tag{2.18e}$$

(ii) Define an ideal $\mathcal{I}_\Lambda$ of the path algebra $k\mathbb{T}_A$ by

$$\mathcal{I}_\Lambda := \left\langle \sum_{i=1}^{r} \lambda_i^{(1)} f_{i,1} f_{i,1^*}, \sum_{i=1}^{r} \lambda_i^{(2)} f_{i,1} f_{i,1^*} \right\rangle, \tag{2.18f}$$

where $(\lambda_1^{(1)}, \lambda_1^{(2)}) = (1,0)$ and $(\lambda_i^{(1)}, \lambda_i^{(2)}) = (\lambda_i, 1)$ for $i = 2, \dots, r$.

We denote by $\widetilde{\mathbb{T}}_{A,\Lambda}$ the bound quiver $(\widetilde{\mathbb{T}}_A, \mathcal{I}_\Lambda)$ for simplicity. The bound quiver algebra $k\widetilde{\mathbb{T}}_{A,\Lambda}$ is called the *octopus* of type (A, Λ).



Remark 2.22. In [8], Clawley-Boevey defines a bound quiver algebra associated to (A, Λ), which is called the *squid.* A squid and an octopus are different but very similar, more precisely, these algebras are not isomorphic but derived equivalent.

2.5. **Algebro-geometric aspect of octopuses.** We associate to a pair (A, Λ) an algebro-geometric object following Geigle–Lenzing (cf. Section 1.1 in [9]).

Definition 2.23. Let $r \geq 3$ be a positive integer, $A = (a_1, \dots, a_r)$ an r-tuple of positive integers greater than one and $\Lambda = (\lambda_1, \dots, \lambda_r)$ an r-tuple of pairwise distinct elements of $\mathbb{P}^1(k)$ normalized such that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$.

(i) Define a ring $S_{A,\Lambda}$ by

$$S_{A,\Lambda} := k[X_1, \dots, X_r] \, / (X_i^{a_i} - X_2^{a_2} + \lambda_i X_1^{a_1}; i = 3, \dots, r) \, . \tag{2.19}$$

(ii) Denote by L_A an abelian group generated by r-letters $\vec{X}_i$, $i = 1, \dots, r$ defined as the quotient

$$L_A := \bigoplus_{i=1}^{r} \mathbb{Z}\vec{X}_i \Big/ (a_i\vec{X}_i - a_j\vec{X}_j; 1 \leq i < j \leq r) \, . \tag{2.20}$$

Note that $S_{A,\Lambda}$ is naturally graded with respect to L_A. Denote by $\mathrm{gr}^{L_A}\text{-}S_{A,\Lambda}$ the category of finitely generated L_A-graded $S_{A,\Lambda}$-modules and by $\mathrm{tor}^{L_A}\text{-}S_{A,\Lambda}$ the full subcategory of $\mathrm{gr}^{L_A}\text{-}S_{A,\Lambda}$ consisting of modules of finite length.

Definition 2.24. Define a stack $\mathbb{P}^1_{A,\Lambda}$ by

$$\mathbb{P}^1_{A,\Lambda} := \left[\left(\mathrm{Spec}(S_{A,\Lambda}) \backslash \{0\} \right) / \mathrm{Spec}(kL_A) \right] , \tag{2.21}$$

which is called the *orbifold projective line* of type (A, Λ). Denote by $\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})$ the category of coherent sheaves on $\mathbb{P}^1_{A,\Lambda}$ and by $\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})$ its bounded derived category.

Properties of categories $\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})$ and $\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})$ are extensively studied by Geigle–Lenzing [9]. Among them, the following is of our interest in this paper.

Proposition 2.25 (Proposition 4.1 in [9]). *There exists an equivalence of triangulated categories*

$$\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda}) \simeq \mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda}). \tag{2.22}$$

2.6. **Generalized root systems associated to octopuses.** Since the assumptions of Proposition 2.9 are proven for $\mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})$ by Meltzer [21], we obtain a generalized root system.

Definition 2.26. Let $k\widetilde{\mathbb{T}}_{A,\Lambda}$ be an octopus of type (A, Λ).

(i) Denote by $\widetilde{R}_A$ the generalized root system associated to $\mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda})$.

(ii) For any $v \in \widetilde{\mathbb{T}}_{A,0}$, denote by P_v the corresponding indecomposable projective $k\widetilde{\mathbb{T}}_{A,\Lambda}$-module, which satisfies $k\widetilde{\mathbb{T}}_{A,\Lambda} = \bigoplus_{v \in \widetilde{T}_A} P_v$ as a $k\widetilde{\mathbb{T}}_{A,\Lambda}$-module. Note that the collection $(P_v)_{v \in \widetilde{T}_A}$ forms a full strongly exceptional collection in $\mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda})$.

(iii) For any $v \in \widetilde{\mathbb{T}}_{A,0}$, denote by S_v the corresponding simple $k\widetilde{\mathbb{T}}_{A,\Lambda}$-module. Note that the collection $(S_v)_{v \in \widetilde{T}_A}$ forms a full exceptional collection in $\mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda})$ such that

$$\chi_{\mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda})}([P_v],[S_{v'}]) = \delta_{vv'}, \quad v, v' \in \widetilde{\mathbb{T}}_{A,0}, \tag{2.23}$$

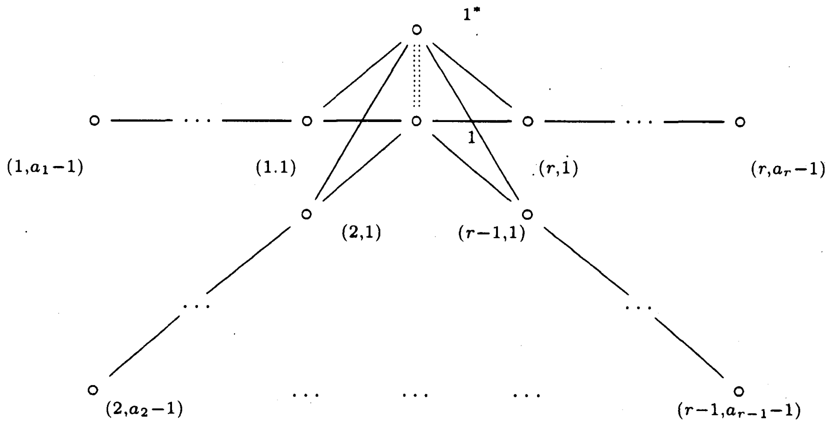
where $\delta_{vv'}$ denotes the Kronecker's delta.

(iv) For any simple $k\widetilde{\mathbb{T}}_{A,\Lambda}$-module S_v, $v \in \widetilde{\mathbb{T}}_{A,0}$, denote by $\widetilde{\alpha}_v$ the equivalence class $[S_v] \in K_0(\widetilde{R}_A) = K_0(\mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda}))$. Set

$$B_{\widetilde{\mathbb{T}}_{A,\Lambda}} := \{\widetilde{\alpha}_v\}_{v \in \widetilde{\mathbb{T}}_{A,0}}, \tag{2.24}$$

which is a root basis of $\widetilde{R}_A$.

(v) Denote by $\widetilde{T}_A$ the Coxeter–Dynkin diagram $\Gamma_{B_{\widetilde{\mathbb{T}}_{A,\Lambda}}}$, which turns out to be the following diagram by using the property (2.23):



We often write $v \in \widetilde{T}_A$ instead of $v \in \widetilde{\mathbb{T}}_{A,0}$.

(vi) For each $v \in \widetilde{T}_A$, define the *simple reflection* $\widetilde{r}_v$ on $K_0(\widetilde{R}_A)$ by

$$\widetilde{r}_v(\widetilde{\lambda}) := \widetilde{\lambda} - I_{\widetilde{R}_A}(\widetilde{\lambda}, \widetilde{\alpha}_v)\widetilde{\alpha}_v, \quad \widetilde{\lambda} \in K_0(\widetilde{R}_A). \tag{2.25}$$

Since $B_{\widetilde{\mathbb{T}}_A}$ is a root basis of $\widetilde{R}_A$, the Weyl group $W(\widetilde{R}_A)$ of $\widetilde{R}_A$ is generated by simple reflections;

$$W(\widetilde{R}_A) = \langle \widetilde{r}_v \mid v \in \widetilde{T}_A \rangle. \tag{2.26}$$

2.7. **A relation between octopuses and star quivers.** Set $\delta := \widetilde{\alpha}_{1^*} - \widetilde{\alpha}_1$. It is easy to see that δ belongs to the radical of the Cartan form $I_{\widetilde{R}_A}$ on $K_0(\widetilde{R}_A)$, therefore the natural projection map

$$K_0(\widetilde{R}_A) \longrightarrow K_0(\widetilde{R}_A)/\mathbb{Z}\delta \cong K_0(R_A) \tag{2.27}$$

induces the surjective group homomorphism

$$p : W(\widetilde{R}_A) \twoheadrightarrow W(R_A). \tag{2.28}$$

Indeed, we have

$$p(\widetilde{r}_1) = p(\widetilde{r}_{1^*}) = r_1, \tag{2.29a}$$

$$p(\widetilde{r}_v) = r_v, \quad v \in \mathbb{T}_{A,0}. \tag{2.29b}$$

Moreover, the correspondence $\alpha_v \mapsto \widetilde{\alpha}_v$ for $v \in T_A$ gives the splitting of the surjective map (2.27) and induces the isomorphism of $\mathbb{Z}$-modules

$$K_0(\widetilde{R}_A) \cong K_0(R_A) \oplus \mathbb{Z}\delta, \tag{2.30}$$

which is compatible with the Cartan forms $I_{\widetilde{R}_A}$ and I_{R_A}. Hence we obtain the group homomorphism

$$i : W(R_A) \longrightarrow W(\widetilde{R}_A), \quad r_v \mapsto \widetilde{r}_v \tag{2.31}$$

such that $p \circ i = \mathrm{id}_{W(R_A)}$.

3. Presentations of Weyl groups

In this section, we describe the Weyl group $W(\widetilde{R}_A)$ as the "affinization" of the Weyl group $W(R_A)$. Lemmas, Propositions and Theorem in this section can be obtained by elementary calculations. For precise proofs, see [31].

Definition 3.1. For each vertex $v \in T_A$, define an element $\widetilde{\tau}_v \in W(\widetilde{R}_A)$ by induction as follows:

- For the vertex 1, set

$$\widetilde{\tau}_1 := \widetilde{r}_1\widetilde{r}_{1^*}. \tag{3.1a}$$

- Set

$$\widetilde{\tau}_{(i,1)} := \widetilde{r}_{(i,1)}\widetilde{\tau}_1\widetilde{r}_{(i,1)}\widetilde{\tau}_1^{-1}, \quad i = 1, \dots, r, \tag{3.1b}$$

$$\widetilde{\tau}_{(i,j)} := \widetilde{r}_{(i,j)}\widetilde{\tau}_{(i,j-1)}\widetilde{r}_{(i,j)}\widetilde{\tau}_{(i,j-1)}^{-1}, \quad i = 1, \dots, r, \ j = 2, \dots, a_i - 1. \tag{3.1c}$$

Denote by N the smallest normal subgroup of $W(\widetilde{R}_A)$ containing $\widetilde{\tau}_1$.

Lemma 3.2. *For all $v \in T_A$, the element $\widetilde{\tau}_v$ belongs to N.*

Proposition 3.3. *For all $v \in T_A$, we have*

$$\widetilde{r}_v(\widetilde{\lambda}) = \widetilde{\lambda} - I_{\widetilde{R}_A}(\widetilde{\lambda}, \widetilde{\alpha}_v)\delta, \quad \widetilde{\lambda} \in K_0(\widetilde{R}_A). \tag{3.2}$$

In particular, there is a natural surjective group homomorphism

$$\varphi : K_0(R_A) \twoheadrightarrow N, \quad \sum_{v \in T_A} m_v \alpha_v \mapsto \prod_{v \in T_A} \widetilde{r}_v^{\,m_v}, \tag{3.3}$$

which induces an isomorphism

$$K_0(R_A)/\mathrm{rad}(I_{R_A}) \cong N. \tag{3.4}$$

Note that $\mathrm{rad}(I_{R_A})$ is zero if $\chi_A \neq 0$ and is of rank one if $\chi_A = 0$.

Proposition 3.4. *For v, $v' \in T_A$, we have*

$$\widetilde{r}_v \widetilde{\tau}_v \widetilde{r}_v = \widetilde{\tau}_v^{-1}, \tag{3.5a}$$

$$\widetilde{r}_v \widetilde{\tau}_{v'} \widetilde{r}_v = \widetilde{\tau}_{v'}, \quad \textit{if} \quad I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = 0, \tag{3.5b}$$

$$\widetilde{r}_v \widetilde{\tau}_{v'} \widetilde{r}_v = \widetilde{\tau}_v \widetilde{\tau}_{v'}, \quad \textit{if} \quad I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = -1. \tag{3.5c}$$

Since the Weyl group $W(R_A)$ is a subgroup of $\mathrm{Aut}(K_0(R_A), I_{R_A})$, we can consider the group $W(R_A) \ltimes K_0(R_A)$, the semi-direct product of $W(R_A)$ and $K_0(R_A)$. Note that the equations (3.5a), (3.5b) and (3.5c) can be thought of as the adjoint action of $W(R_A)$ on the free generators of $K_0(R_A)$ expressed in multiplicative notation since we have $\widetilde{r}_v(\widetilde{\alpha}_v) = -\widetilde{\alpha}_v$, $\widetilde{r}_v(\widetilde{\alpha}_{v'}) = \widetilde{\alpha}_{v'}$ if $I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = 0$ and $\widetilde{r}_v(\widetilde{\alpha}_{v'}) = \widetilde{\alpha}_v + \widetilde{\alpha}_{v'}$ if $I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = -1$.

Moreover, since the Weyl group $W(R_A)$ respects the radical $\mathrm{rad}(I_{R_A})$, we can also consider the group $W(R_A) \ltimes (K_0(R_A)/\mathrm{rad}(I_{R_A}))$, the semi-direct product of $W(R_A)$ and $K_0(R_A)/\mathrm{rad}(I_{R_A})$, which is isomorphic to $W(\widetilde{R}_A)$. More precisely, we have the following.

Theorem 3.5. *There is an exact sequence of groups*

$$\{1\} \longrightarrow N \longrightarrow W(\widetilde{R}_A) \xrightarrow{p} W(R_A) \longrightarrow \{1\}. \tag{3.6}$$

In particular, we have an isomorphism

$$W(\widetilde{R}_A) \cong W(R_A) \ltimes (K_0(R_A)/\mathrm{rad}(I_{R_A}))\,. \tag{3.7}$$

Therefore it turns out that $W(\widetilde{R}_A)$ is an affine Weyl group if $\chi_A > 0$.

Definition 3.6. Let the notations be as above.

(i) If $\chi_A < 0$, then the group $W(\widetilde{R}_A)$ is called the *cuspidal Weyl group* of type A, which is isomorphic to $W(R_A) \ltimes K_0(R_A)$ by Theorem 3.5.

(ii) If $\chi_A = 0$, then the group $W(\widetilde{R}_A)$ is called the *elliptic Weyl group* of type A.

(iii) If $\chi_A = 0$, then the group $W(R_A) \ltimes K_0(R_A)$ is isomorphic to the non-trivial central extension of $W(\widetilde{R}_A)$ by $\mathbb{Z}$, which is called the *hyperbolic extension* of the elliptic Weyl group $W(\widetilde{R}_A)$ (cf. Section 1.18 in [25]).

4. Weyl groups as generalized Coxeter groups

In this section, we express the Weyl group $W(\widetilde{R}_A)$ as a generalized Coxeter group. Lemmas, Propositions and Theorem in this section can be also obtained by elementary calculations. For precise proofs, see [31]. First we note the following fact.

Proposition 4.1. *Define a group $W(T_A)$ by the following generators and the Coxeter relations attached to the diagram T_A :*

Generators: $\{w_v \mid v \in T_A\}$

Relations:

$$w_v^2 = 1 \quad \text{for all} \quad v \in T_A, \tag{4.1a}$$

$$w_v w_{v'} = w_{v'} w_v \quad \text{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = 0, \tag{4.1b}$$

$$w_v w_{v'} w_v = w_{v'} w_v w_{v'} \quad \text{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = -1. \tag{4.1c}$$

Then the correspondence $w_v \mapsto r_v$ for $v \in T_A$ induces an isomorphism of groups

$$W(T_A) \cong W(R_A). \tag{4.2}$$

Proposition 4.2. *Define a group $W(T_A) \ltimes K_0(R_A)$ by the following generators and the relations:*

Generators: $\{w_v, \tau_v \mid v \in T_A\}$

Relations:

$$w_v^2 = 1 \quad \text{for all} \quad v \in T_A, \tag{4.3a}$$

$$w_v w_{v'} = w_{v'} w_v \quad \text{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = 0, \tag{4.3b}$$

$$w_v w_{v'} w_v = w_{v'} w_v w_{v'} \quad \text{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = -1, \tag{4.3c}$$

$$\tau_v \tau_{v'} = \tau_{v'} \tau_v \quad \text{for all} \quad v, v' \in T_A, \tag{4.3d}$$

$$w_v \tau_v w_v = \tau_v^{-1} \quad \text{for all} \quad v \in T_A, \tag{4.3e}$$

$$w_v \tau_{v'} = \tau_{v'} w_v \quad \text{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = 0, \tag{4.3f}$$

$$w_v \tau_{v'} w_v = \tau_{v'} \tau_v \quad \text{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = -1. \tag{4.3g}$$

Identify the subgroup generated by τ_v, $v \in T_A$ with a free abelian group $K_0(R_A)$ expressed in multiplicative notation.

(i) *The correspondence $w_v \mapsto r_v$, $\tau_v \mapsto \tau_v$ for $v \in T_A$ induces an isomorphism of groups*

$$W(T_A) \ltimes K_0(R_A) \cong W(R_A) \ltimes K_0(R_A), \tag{4.4}$$

where the semi-direct product in the right hand side is given by the natural inclusion $W(R_A) \hookrightarrow \mathrm{Aut}(K_0(R_A), I_{R_A})$.

(ii) *The correspondence $w_v \mapsto r_v$, $\tau_v \mapsto \widetilde{\tau}_v$ for $v \in T_A$ induces a surjective group homomorphism*

$$W(T_A) \ltimes K_0(R_A) \twoheadrightarrow W(\widetilde{R}_A), \tag{4.5}$$

whose kernel is isomorphic to $\mathrm{rad}(I_{R_A})$.

Definition 4.3. Define a group $W(\widetilde{T}_A)$ by the following generators and the generalized Coxeter relations attached to the diagram $\widetilde{T}_A$:

Generators: $\{\widetilde{w}_v \mid v \in \widetilde{T}_A\}$

Relations:

$$\widetilde{w}_v^2 = 1 \quad \text{for all} \quad v \in \widetilde{T}_A, \tag{W0}$$

$$\widetilde{w}_v\widetilde{w}_{v'} = \widetilde{w}_{v'}\widetilde{w}_v \quad \text{if} \quad I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = 0, \tag{W1.0}$$

$$\widetilde{w}_v\widetilde{w}_{v'}\widetilde{w}_v = \widetilde{w}_{v'}\widetilde{w}_v\widetilde{w}_{v'} \quad \text{if} \quad I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = -1, \tag{W1.1}$$

$$\widetilde{w}_{(i,1)}\sigma_1\widetilde{w}_{(i,1)}\sigma_1 = \sigma_1\widetilde{w}_{(i,1)}\sigma_1\widetilde{w}_{(i,1)}, \tag{W2}$$

$$\begin{cases} \widetilde{w}_{(i,1)}\sigma_{(j,1)} = \sigma_{(j,1)}\widetilde{w}_{(i,1)} \\ \widetilde{w}_{(j,1)}\sigma_{(i,1)} = \sigma_{(i,1)}\widetilde{w}_{(j,1)} \end{cases} \quad \text{for all} \quad 1 \le i < j \le r, \tag{W3}$$

where $\sigma_1 := \widetilde{w}_1\widetilde{w}_{1^*}$ and $\sigma_{(i,1)} := \widetilde{w}_{(i,1)}\sigma_1\widetilde{w}_{(i,1)}\sigma_1^{-1}$ for all $i = 1, \ldots, r$.

The conditions (**W2**) and (**W3**) are different from the definition in [28]. However we can deduce the original ones from (**W2**) and (**W3**) under the conditon (**W0**):

Proposition 4.4 (cf. Lemma 4.1 and Lemma 4.2 in [33]). *Under the relation* **W0**, *we have the following equivalences of relations*:

$$\widetilde{w}_{(i,1)}\sigma_1\widetilde{w}_{(i,1)}\sigma_1 = \sigma_1\widetilde{w}_{(i,1)}\sigma_1\widetilde{w}_{(i,1)} \iff (\widetilde{w}_1\widetilde{w}_{(i,1)}\widetilde{w}_{1^*}\widetilde{w}_{(i,1)})^3 = 1, \tag{4.7a}$$

$$\widetilde{w}_{(i,1)}\sigma_{(j,1)} = \sigma_{(j,1)}\widetilde{w}_{(i,1)} \iff (\widetilde{w}_{(i,1)}\widetilde{w}_1\widetilde{w}_{(i,1)}\widetilde{w}_{1^*}\widetilde{w}_{(j,1)}\widetilde{w}_{1^*})^2 = 1, \tag{4.7b}$$

$$\widetilde{w}_{(j,1)}\sigma_{(i,1)} = \sigma_{(i,1)}\widetilde{w}_{(j,1)} \iff (\widetilde{w}_{(i,1)}\widetilde{w}_{1^*}\widetilde{w}_{(i,1)}\widetilde{w}_1\widetilde{w}_{(j,1)}\widetilde{w}_1)^2 = 1. \tag{4.7c}$$

Note that the Coxeter–Dynkin diagram $\widetilde{T}_A$ is symmetric under the permutation

$$1^* \mapsto 1, \ 1 \mapsto 1^*, \ v \mapsto v \quad \text{if} \quad v \neq 1, 1^*. \tag{4.8}$$

This symmetry of $\widetilde{T}_A$ induces the automorphism on $W(\widetilde{R}_A)$ which sends σ_1 to σ_1^{-1} and hence $W(\widetilde{T}_A)$ depends only on the Coxeter–Dynkin diagram $\widetilde{T}_A$.

Theorem 4.5. *We have an isomorphism of groups*

$$W(\widetilde{T}_A) \cong W(R_A) \ltimes K_0(R_A). \tag{4.9}$$

In particular, $W(\widetilde{T}_A) \cong W(\widetilde{R}_A)$ *if* $\chi_A \neq 0$ *and* $W(\widetilde{T}_A)$ *is isomorphic to the hyperbolic extension of the elliptic Weyl group* $W(\widetilde{R}_A)$ *if* $\chi_A = 0$.

5. Cuspidal Artin Groups

In this section, we obtain a relation between the generalized Coxeter group $W(\widetilde{T}_A)$ and the fundamental group of regular orbit space for $W(R_A) \ltimes K_0(R_A)$.

Definition 5.1. Define a group $G(\widetilde{T}_A)$ by the following generators and the generalized Coxeter relations attached to the diagram $\widetilde{T}_A$:

Generators: $\{\widetilde{g}_v \mid v \in \widetilde{T}_A\}$

Relations:

$$\widetilde{g}_v\widetilde{g}_{v'} = \widetilde{g}_{v'}\widetilde{g}_v \quad \text{if} \quad I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = 0, \tag{\textbf{A1.0}}$$

$$\widetilde{g}_v\widetilde{g}_{v'}\widetilde{g}_v = \widetilde{g}_{v'}\widetilde{g}_v\widetilde{g}_{v'} \quad \text{if} \quad I_{\widetilde{R}_A}(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = -1, \tag{\textbf{A1.1}}$$

$$\widetilde{g}_{(i,1)}\widetilde{\rho}_1\widetilde{g}_{(i,1)}\widetilde{\rho}_1 = \widetilde{\rho}_1\widetilde{g}_{(i,1)}\widetilde{\rho}_1\widetilde{g}_{(i,1)} \quad \text{for all } i = 1, \ldots, r, \tag{\textbf{A2}}$$

$$\begin{cases} \widetilde{g}_{(i,1)}\widetilde{\rho}_{(j,1)} = \widetilde{\rho}_{(j,1)}\widetilde{g}_{(i,1)} \\ \widetilde{g}_{(j,1)}\widetilde{\rho}_{(i,1)} = \widetilde{\rho}_{(i,1)}\widetilde{g}_{(j,1)} \end{cases} \quad \text{for all} \quad 1 \leq i < j \leq r. \tag{\textbf{A3}}$$

where $\widetilde{\rho}_1 := \widetilde{g}_1\widetilde{g}_{1^*}$ and $\widetilde{\rho}_{(i,1)} := \widetilde{g}_{(i,1)}\widetilde{\rho}_1\widetilde{g}_{(i,1)}\widetilde{\rho}_1^{-1}$ for all $i = 1, \ldots, r$.

Definition 5.2. Let the notations be as above.

(i) If $\chi_A = 0$, then the group $G(\widetilde{T}_A)$ is called the *elliptic Artin group* of type A.

(ii) If $\chi_A < 0$, then the group $G(\widetilde{T}_A)$ is called the *cuspidal Artin group* of type A.

Remark 5.3. It turns out later that the group $G(\widetilde{T}_A)$ is an affine Artin group by Theorem 5.8 if $\chi_A > 0$.

In addition to $\widetilde{\rho}_1$, $\widetilde{\rho}_{(i,1)}$, we also define the element $\widetilde{\rho}_{(i,j+1)}$ inductively as follows:

$$\widetilde{\rho}_{(i,j+1)} := \widetilde{g}_{(i,j+1)}\widetilde{\rho}_{(i,j)}\widetilde{g}_{(i,j+1)}\widetilde{\rho}_{(i,j)}^{-1}, \quad i = 1, \ldots, r, \ j = 1, \ldots, a_i - 2. \tag{5.2}$$

The following proposition is obvious from Definition 4.3:

Proposition 5.4. *The correspondence* $\widetilde{g}_v \mapsto \widetilde{w}_v$ *for* $v \in \widetilde{T}_A$ *induces a surjective group homomorphism*

$$G(\widetilde{T}_A) \twoheadrightarrow W(\widetilde{T}_A), \tag{5.3}$$

which yields an isomorphism

$$G(\widetilde{T}_A) \Big/ \langle \widetilde{g}_v^2 \mid v \in \widetilde{T}_A \rangle \cong W(\widetilde{T}_A). \tag{5.4}$$

Definition 5.5. Define a complex manifold $\mathcal{E}(R_A)$ by

$$\mathcal{E}(R_A) := \{h \in K_0(R_A)^*_{\mathbb{C}} \mid \mathrm{Im}(h) \in C(R_A)\}, \tag{5.5}$$

where $C(R_A)$ is the topological interior of the *Tits cone* $\overline{C}(R_A)$ of R_A :

$$\overline{C}(R_A) := \bigcup_{w \in W(R_A)} w\left(\{h \in K_0(R_A)^*_{\mathbb{C}} \mid h(\alpha_v) \geq 0, \text{ for all } v \in T_A\}\right). \tag{5.6}$$

Set

$$\mathcal{E}(R_A)^{reg} := \mathcal{E}(R_A) \setminus \bigcup_{\alpha \in \Delta_{re}(R_A), n \in \mathbb{Z}} H_{\alpha,n}. \tag{5.7}$$

where we denote by $H_{\alpha,n}$ the reflection hyperplane associated to $\widetilde{T}_A$, i.e.,

$$H_{\alpha,n} := \{h \in K_0(R_A)^*_{\mathbb{C}} \mid h(\alpha) = n\}. \tag{5.8}$$

The group $W(R_A) \ltimes K_0(R_A)$ naturally acts on $\mathcal{E}(R_A)$ in a properly discontinuous way. It is known that the action is free on $\mathcal{E}(R_A)^{reg}$.

Definition 5.6. Define a group $G(\widetilde{R}_A)$ as the fundamental group of the regular orbit space:

$$G(\widetilde{R}_A) := \pi_1(\mathcal{E}(R_A)^{reg}/(W(R_A) \ltimes K_0(R_A)), *). \tag{5.9}$$

Remark 5.7. Since the complex manifold $\mathcal{E}(R_A)^{reg}$ is connected, the group $G(\widetilde{R}_A)$ does not depend on the base point $*$.

By definition of fundamental groups, we have the following commutative diagram of groups:

$$\begin{array}{ccccccccc}
 & & & & \{1\} & & \{1\} & & \\
 & & & & \downarrow & & \downarrow & & \\
\{1\} & \longrightarrow & \pi_1(\mathcal{E}(R_A)^{reg}, *) & \longrightarrow & \pi_1(\mathcal{E}(R_A)^{reg}/K_0(R_A), *) & \longrightarrow & K_0(R_A) & \longrightarrow & \{1\} \\
 & & \| & & \downarrow & & \downarrow & & \\
\{1\} & \longrightarrow & \pi_1(\mathcal{E}(R_A)^{reg}, *) & \longrightarrow & G(\widetilde{R}_A) & \longrightarrow & W(R_A) \ltimes K_0(R_A) & \longrightarrow & \{1\} \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W(R_A) & = & W(R_A) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \{1\} & & \{1\} & &
\end{array}$$

Generalizing the result for $\chi_A = 0$ by Yamada [33], we obtain the following:

Theorem 5.8. *There exists an isomorphism of groups*

$$G(\widetilde{T}_A) \cong G(\widetilde{R}_A). \tag{5.10}$$

Sketch of Proof. We can obtain the natural surjective homomorphism from $G(\widetilde{R}_A)$ to $G(\widetilde{T}_A)$ by using the following description of $G(\widetilde{R}_A)$ by Van der Lek [32]:

Proposition 5.9 ([32]). *The group $G(\widetilde{R}_A)$ is described by the following generators and relations:*

Generators: $\{g_v, \rho_v \mid v \in T_A\}$

Relations:

$$g_v g_{v'} = g_{v'} g_v \quad \textit{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = 0, \tag{5.11a}$$

$$g_v g_{v'} g_v = g_{v'} g_v g_{v'} \quad \textit{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = -1, \tag{5.11b}$$

$$\rho_v \rho_{v'} = \rho_{v'} \rho_v \quad \textit{for all} \quad v, v' \in T_A, \tag{5.11c}$$

$$g_v \rho_{v'} = \rho_{v'} g_v \quad \textit{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = 0, \tag{5.11d}$$

$$g_v \rho_{v'} g_v = \rho_{v'} \rho_v \quad \textit{if} \quad I_{R_A}(\alpha_v, \alpha_{v'}) = -1. \tag{5.11e}$$

We can construct the inverse homomorphism by putting the element g_{1^*} of the group $G(\widetilde{R}_A)$ by $g_{1^*} := g_1^{-1}\rho_1$. This argument is exactly the same as in Yamada [33]. □

The following corollary is obvious from Proposition 4.2 and Proposition 5.9:

Corollary 5.10. *The correspondences $g_v \mapsto w_v$, $\rho_v \mapsto \tau_v$ for $v \in T_A$ induces a surjective group homomorphism*

$$G(\widetilde{R}_A) \twoheadrightarrow W(T_A) \ltimes K_0(T_A), \tag{5.12}$$

which yields an isomorphism

$$G(\widetilde{R}_A) \Big/ \langle g_v^2,\ g_v\rho_v g_v \rho_v \mid v \in \widetilde{T}_A \rangle \cong W(R_A) \ltimes K_0(R_A). \tag{5.13}$$

There exists the following commutative diagram of groups

$$\begin{array}{ccc} G(\widetilde{T}_A) & \longrightarrow & G(\widetilde{R}_A) \\ \downarrow & & \downarrow \\ W(\widetilde{T}_A) & \longrightarrow & W(R_A) \ltimes K_0(R_A) \end{array}, \tag{5.14}$$

where the upper horizontal homomorphism is the isomorphisms in Theorem 4.5, the lower horizontal homomorphism is the isomorphisms in Theorem 5.8, the left vertical homomorphisms is the one in Proposition 5.4 and finally, the right vertical homomorphism is the one in Corollary 5.10.

6. Autoequivalence group

In this section, we compare the cuspidal Artin group $G(\widetilde{T}_A)$ with a subgroup of autoequivalence group for the derived category of the 2-Calabi–Yau completion of $k\widetilde{\mathbb{T}}_{A,\Lambda}$ generated by some spherical twist functors.

Definition 6.1. Put $\mathcal{A} := k\widetilde{\mathbb{T}}_{A,\Lambda}$ and consider it as a dg k-algebra concentrated in the degree 0. Let $\Theta_{\mathcal{A}}$ be the cofibrant replacement of the complex $\mathbb{R}\mathrm{Hom}_{\mathcal{A}\otimes_k\mathcal{A}^{op}}(\mathcal{A}, \mathcal{A}\otimes_k\mathcal{A}^{op})$. The 2-*Calabi–Yau completion* (or *derived* 2-*preprojective algebra*) of $\mathcal{A}$ is the following tensor dg k-algebra:

$$\Pi_2(\mathcal{A}) := \mathcal{A}\bigoplus_{n\in\mathbb{N}}\underbrace{(\Theta_{\mathcal{A}}[1]\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\Theta_{\mathcal{A}}[1])}_{n-times}. \tag{6.1}$$

Remark 6.2. Since $k\widetilde{\mathbb{T}}_{A,\Lambda}$ is a directed finite dimensional algebra over the *field* k of global dimension two, the above definition agrees with the original one in [17].

Let $\mathcal{D}(\Pi_2(\mathcal{A}))$ be the derived category of dg $\Pi_2(\mathcal{A})$-modules. Note that we have a natural functor $\mathcal{D}(k\widetilde{\mathbb{T}}_{A,\Lambda}) \longrightarrow \mathcal{D}(\Pi_2(\mathcal{A}))$ given by the restriction along the projection onto the first component $\Pi_2(\mathcal{A}) \longrightarrow \mathcal{A} = k\widetilde{\mathbb{T}}_{A,\Lambda}$. Therefore we shall often regard $M \in \mathcal{D}(k\widetilde{\mathbb{T}}_{A,\Lambda})$ also as a dg $\Pi_2(\mathcal{A})$-module.

Let $\check{\mathcal{D}}_{A,\Lambda}$ be the smallest full triangulated subcategory of $\mathcal{D}(\Pi_2(\mathcal{A}))$ containing $k\widetilde{\mathbb{T}}_{A,\Lambda}$, closed under isomorphisms and taking direct summand. By the definition of $\check{\mathcal{D}}_{A,\Lambda}$, we have the following proposition:

Proposition 6.3. *The functor $\mathcal{D}(k\widetilde{\mathbb{T}}_{A,\Lambda}) \longrightarrow \mathcal{D}(\Pi_2(\mathcal{A}))$ induces an isomorphism of abelian groups $K_0(\widetilde{R}_A) = K_0(\mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda})) \cong K_0(\check{\mathcal{D}}_{A,\Lambda})$.*

Proposition 6.4 (Lemma 4.4 b) in [17]). *For any $X, Y \in \mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda})$, there is a canonical isomorphism in $\mathcal{D}^b(k)$:*

$$\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(X,Y) \cong \mathbb{R}\mathrm{Hom}_{\mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda})}(X,Y) \oplus \mathbb{R}\mathrm{Hom}_{\mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda})}(Y,X)^*[-2]. \tag{6.2}$$

Corollary 6.5. *Under the isomorphism $K_0(\widetilde{R}_A) \cong K_0(\check{\mathcal{D}}_{A,\Lambda})$ in Proposition 6.3, the Euler form $\chi_{\check{\mathcal{D}}_{A,\Lambda}}$ is identified with the Cartan form $I_{\mathcal{D}^b(k\widetilde{\mathbb{T}}_{A,\Lambda})}$.*

Recall the definitions of spherical objects and spherical twist functors and their properties in Seidel–Thomas [30].

Definition 6.6. An object $S \in \check{\mathcal{D}}_{A,\Lambda}$ is called a 2-*spherical object* if the following conditions are satisfied:

(i) There exists an isomorphism in $\mathcal{D}^b(k)$:

$$\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(S,S) \cong k \oplus k[-2] \tag{6.3}$$

(ii) For all $X \in \check{\mathcal{D}}_{A,\Lambda}$, the composition induces the following perfect pairing:

$$\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(X, S[2]) \otimes_k \mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(S, X) \to \mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(S, S[2]) \cong k. \tag{6.4}$$

Definition 6.7. Let S be a spherical object in $\check{\mathcal{D}}_{A,\Lambda}$ and X an object in $\check{\mathcal{D}}_{A,\Lambda}$. Define $T_S X \in \check{\mathcal{D}}_{A,\Lambda}$ by the cone of the evaluation morphism ev

$$\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(S, X) \otimes^{\mathbb{L}} S \xrightarrow{ev} X. \tag{6.5}$$

Similarly, define $T_S^- X \in \check{\mathcal{D}}_{A,\Lambda}$ by the -1-translation of the cone of the evaluation morphism ev^*

$$X \xrightarrow{ev^*} \mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(X, S)^* \otimes^{\mathbb{L}} S. \tag{6.6}$$

The operations T_S and T_S^- define endo-functors on $\check{\mathcal{D}}_{A,\Lambda}$, which are called the *spherical twist* functors.

We collect some basic properties of the spherical twist functors. In particular, it turns out that the spherical twist functors are autoequivalences on $\check{\mathcal{D}}_{A,\Lambda}$.

Proposition 6.8 (Proposition 2.10, Lemma 2.11, Proposition 2.13 in [30]). *Let S be a spherical object in $\check{\mathcal{D}}_{A,\Lambda}$.*

(i) *For an integer $i \in \mathbb{Z}$, we have $T_{S[i]} \cong T_S$.*

(ii) *We have $T_S^- T_S \cong \mathrm{Id}_{\check{\mathcal{D}}_{A,\Lambda}}$ and $T_S T_S^- \cong \mathrm{Id}_{\check{\mathcal{D}}_{A,\Lambda}}$.*

(iii) *We have $T_S S \cong S[-1]$.*

(iv) *For any spherical object S', we have*

$$T_S T_{S'} \cong T_{T_S S'} T_S. \tag{6.7}$$

(v) *For any spherical object S' such that $\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(S', S) \cong k[-1]$ in $\mathcal{D}(k)$, we have an isomorphism*

$$T_S T_{S'} S \cong S' \quad \text{in} \quad \check{\mathcal{D}}_{A,\Lambda}. \tag{6.8}$$

Recall that S_v is the simple $k\widetilde{\mathbb{T}}_{A,\Lambda}$-module corresponding to the vertex $v \in \widetilde{T}_A$ (see Definition 2.26), which we regard as a dg $\Pi_2(k\widetilde{\mathbb{T}}_{A,\Lambda})$-module. The following two propositions hold from Propositon 6.4:

Proposition 6.9. *For any $v \in \widetilde{T}_A$, S_v is a spherical object in $\check{\mathcal{D}}_{A,\Lambda}$.*

Proposition 6.10. *Under the isomorphism $K_0(\check{\mathcal{D}}_{A,\Lambda}) \cong K_0(\widetilde{R}_A)$ in Proposition 6.3, the automorphism of $K_0(\widetilde{R}_A)$ induced by T_{S_v} is identified with the simple reflection $\widetilde{r}_v \in W(\widetilde{R}_A)$.*

Definition 6.11. Denote by $\mathrm{Br}(\check{\mathcal{D}}_{A,\Lambda})$ the subgroup of $\mathrm{Auteq}(\check{\mathcal{D}}_{A,\Lambda})$ generated by the elements T_{S_v} for $v \in \widetilde{T}_A$.

Theorem 6.12. *The correspondence $\widetilde{g}_v \mapsto T_{S_v}$ for $v \in \widetilde{T}_A$ induces a surjective group homomorphism*

$$G(\widetilde{T}_A) \twoheadrightarrow \mathrm{Br}(\check{\mathcal{D}}_{A,\Lambda}). \tag{6.9}$$

Sketch of Proof. Set $T_v := T_{S_v}$ for the simplicity. We only need to check that the elements T_v for $v \in \widetilde{T}_A$ satisfy the relations (**A2**) and (**A3**) since the relations (**A1.0**) and (**A1.1**) are already known by Seidel–Thomas (Theorem 2.17 in [30]). We can show the assertion mentioned above by using the following two lemmas:

Lemma 6.13. *There are the following isomorphisms in $\mathcal{D}^b(k)$:*

$$\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(S_{1^*}, T_{(i,1)}S_1) \cong k[-2], \tag{6.10a}$$

$$\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(T_{(i,1)}S_1, S_{1^*}) \cong k. \tag{6.10b}$$

By this lemma and the equation (6.8), we get

$$T_1T_{1^*}T_{T_{(i,1)}S_1}S_{1^*} \cong T_1T_{(i,1)}S_1[1] \cong S_{(i,1)}[1].$$

Therefore, $T_{T_1T_{1^*}T_{(i,1)}T_1T_{1^*}S_{(i,1)}} \cong T_{(i,1)}$, which gives the relation (**A2**), namely,

$$T_{(i,1)}T_1T_{1^*}T_{(i,1)}T_1T_{1^*} \cong T_1T_{1^*}T_{(i,1)}T_1T_{1^*}T_{(i,1)}. \tag{6.11}$$

Lemma 6.14. *For $1 \leq i < j \leq r$, there are the following isomorphisms in $\mathcal{D}(k)$:*

$$\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(S_{(i,1)}, T_1T_{1^*}S_{(j,1)}) \cong 0, \tag{6.12a}$$

$$\mathbb{R}\mathrm{Hom}_{\check{\mathcal{D}}_{A,\Lambda}}(T_1T_{1^*}S_{(j,1)}, S_{(i,1)}) \cong 0. \tag{6.12b}$$

By this lemma, we get

$$T_{(j,1)}T_{T_1T_{1^*}S_{(j,1)}}S_{(i,1)} \cong T_{(j,1)}S_{(i,1)} \cong S_{(i,1)}.$$

Therefore, we have the relation (**A3**), namely,

$$T_{(i,1)}T_{(j,1)}T_1T_{1^*}T_{(j,1)}T_{1^*}^{-}T_1^{-} \cong T_{(j,1)}T_1T_{1^*}T_{(j,1)}T_{1^*}^{-}T_1^{-}T_{(i,1)}.$$

We have finished the proof of the theorem. □

There exists the following commutative diagram of groups

$$\begin{array}{ccc} G(\widetilde{T}_A) & \longrightarrow & \mathrm{Br}(\check{\mathcal{D}}_{A,\Lambda}) \\ \downarrow & & \downarrow \\ W(\widetilde{T}_A) & \longrightarrow & W(\widetilde{R}_A) \end{array}, \tag{6.13}$$

where the upper horizontal homomorphism is induced by the above correspondence, the lower horizontal homomorphism is the composition of the morphisms in Proposition 4.2 (ii) and Theorem 4.5, the left vertical homomorphism is the surjective one in Theorem 5.4 and finally, the right vertical homomorphism is induced by the correspondence $T_{S_v} \mapsto \widetilde{r}_v$ for $v \in \widetilde{T}_A$. Recall that the lower horizontal homomorphism is an isomorphism when $\chi_A \neq 0$. We expect that if $\chi_A \neq 0$ then the upper horizontal homomorphism is also an isomorphism.

We conclude this report by stating the conjecture related to Theorem 6.12. Similar to the results by Bridgeland for K3 surfaces in [4] and Kleinian singularities in [5], we expect the following conjecture:

Conjecture 6.15. *The group homomorphism $G(\widetilde{T}_A) \twoheadrightarrow \mathrm{Br}(\check{\mathcal{D}}_{A,\Lambda})$ in Theorem 6.12 should also be injective, and hence isomorphism. In other words, the space of stability condition* $\mathrm{Stab}(\check{\mathcal{D}}_{A,\Lambda})$ *should be simply connected.*

Similar known results for the injectivity of the group homomorphism in Conjecture 6.15 are obtained by Brav–Thomas [6], Ishii–Ueda–Uehara [11] and Seidel–Thomas [30]. The above conjecture is a further theme to be worked on.

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